CS 798: Homework Assignment 2 (Probability)

Assigned: September 30, 2009

1.0 Sample space

In the IEEE 802.11 protocol, the congestion window (CW) parameter is used as follows: initially, a terminal waits for a random time period (called *backoff*) chosen in the range $[1, 2^{CW}]$ before sending a packet. If an acknowledgement for the packet is not received in time, then CW is doubled, and the process is repeated, until CW reaches the value CWMAX. The intial value of CW is CWMIN. What is the sample space for (a) the value of CW? (b) the value of the the backoff?

Solution: The sample space for CW is the discrete set {CWMIN, 2* CWMIN, 4* CWMIN, ...2^n*K*CWMIN}, where K is chosen so that 2^n*K*CWMIN < CWMAX. The sample space for backoff, given CW is a subset of the real line defined by [0, CW].

2.0 Interpretations of probability

Consider the statement: given the conditions right now, the probability of a snowstorm tomorrow morning is 25%. How would you interpret this statement from the perspective of an objective, frequentist, and subjective interpretation of probability (assuming these are possible)?

Solution: An objective interpretation would be that we have a complete weather model that has an intrisic source of randomness. Given this model and the current weather conditions, the model predicts that the probablity of a snowstorm is 25%.

A frequentist approach would be to look at all prior days where today's weather conditions also held, and look at the number of such days where there was a snowstorm the next morning. We would see that 25% of the time, given the current weather, there was as snowstorm.

A subjective interpretation would be that an expert, who knew all the variables, would take 4:1 odds (or bettter) on a bet that it would snow tomorrow.

3.0 Conditional probability

Consider a device that samples packets on a link. (a) Suppose that measurements show that 20% of packets are UDP, and that 10% of all packets are UDP packets with a packet size of 100 bytes.What is the conditional probability that a UDP packet has size 100 bytes? (b) Suppose 50% of packets were UDP, and 50% of UDP packets were 100 bytes long. What fraction of all packets are 100 byte UDP packets?

Solution: (a) We have P(UDP) = 0.2, and P(UDP AND 100) = 0.1. So, P(100 | UDP) = 0.1/0.2 = 0.5.

(b) Here, P(UDP) = 0.5 and P(100|UDP) = 0.5. So, P(100 AND UDP) = 0.5*0.5 = 0.25.

4.0 Conditional probability again

Continuing with Ex. 3: How does the knowledge of the protocol type change the sample space of possible packet lengths? In other words, what is the sample space before and after you know the protocol type of a packet?

Solution: Before you know the protocol type of a packet, the sample space is all possible packet lengths of all possible protocol types. After you know the protocol type, the sample space only include packet lengths for that protocol.

5.0 Bayes' rule

For Exercise 3(a), what additional information do you need to compute P(UDP|100)? Setting that value to x, express P(UDP|100) in terms of x.

Solution: P(UDP|100) = (P(100|UDP)P(UDP))/P(100). We need P(100) = x. Then, P(UDP|100) = 0.5*0.2/x = 0.1/x.

6.0 Cumulative distribution function

(a) Suppose discrete random variable D take values {1, 2, 3, ...,i,...} with probability $1/2^i$. What is its CDF?

(b) Suppose continuous random variable C is uniform in the range [x_1, x_2]. Whats is its CDF?

Solution: (a) $F_D(i) = \sum_{j=1}^{i} \frac{1}{2^j} = 1\text{-}2^{i\text{-}1}$.

(b) $f_{C(x)} = \frac{1}{x_2 - x_1}$, so $F_C(x) = \int_{x_1}^{x} \frac{1}{x_2 - x_1} dx = \frac{x - x_1}{x_2 - x_1}$.

7.0 Expectations

Compute the expectations of the D and C in Exercise 6.

Solution: (a) $E[D] = \sum_{j=1}^{i} \frac{i}{2^j}$.

(b) By geometry, E[C] = (x_2+x_1)/2 (you can also derive this analytically).

8.0 Variance

Prove that $V[aX] = a^2V[X]$.

Solution: $V[aX] = E[a^2X^2] - (E[aX])^2 = a^2(E[X^2] - (E[X])^2) = a^2V[X]$.

9.0 Bernoulli distribution

A hotel has 20 guest rooms. Assuming outgoing calls are independent and that a guest room makes 10 minutes worth of outgoing calls during the busiest hour of the day, what is the probability that 5 calls are simultaneously active during the busiest hour? What is the probability of 15 simultaneous calls?

Solution: Consider the event E defined as 'Room X is making an outgoing call during the busy hour.'

Clearly, P(E) =p = 1/6. The probability of 5 simultaneous calls is $\binom{20}{5}\left(\frac{1}{6}^5\right)\left(\frac{5}{6}\right)^{15} = 0.322$ and of 15 simultaneous calls is $\binom{20}{15}\left(\frac{1}{6}^{15}\right)\left(\frac{5}{6}\right)^{5} = 1.41*10^{-8}$.

10.0 Geometric distribution

Consider a link that has a packet loss rate of 10%. Suppose that every packet transmission has to be acknowledged. Compute the expected number of data transmissions for a successful packet+ack transfer.

Solution: Packet and ack transmissions are geometrically distributed with parameter p=0.1. So the expected number of packet transmissions is $1/p$ = 10 and the expected number of ack transmissions is also 10. These are independent events, so the expected number of data transmissions for successful packet+ack transfer = 10+10 = 20.

11.0 Poisson distribution

Consider a binomially distributed random variable X with parameters n=10, p=0.1. (a) Compute the value of $P(X=8)$ using both the binomial distribution and the Poisson approximation. (b) Repeat for n=100, p=0.1

Solution: (a) Using the binomial distribution, the value is $\binom{10}{8}(0.1^8)(0.9^2)$ = .36*10^{-6}. For the Poisson approximation, λ= 1, so the value is $P(X=8) = e^{-1}\left(\frac{1^8}{8!}\right)$ = 8.9*10^{-6}. (b) Using the binomial distribution, the value is $\binom{100}{8}(0.1^8)(0.9^{92})$ = .114. For the Poisson approximation, λ= 10, so the value is $P(X=8) = e^{-10}\left(\frac{10^8}{8!}\right)$ = .112. It is clear that as n increases, the approximation greatly improves.

12.0 Gaussian distribution

Prove that if X is Gaussian with parameters (μ, σ^2), then the random variable $Y=aX+b$, where a and b are constants, is also Gaussian, with parameters$(a\mu+b, (a\sigma)^2)$.

Solution: Consider the cumulative distribution of $Y = F_Y(y)$ =

$P(Y \le y) = P(aX+b \le y) = P\left(X \le \frac{(y-b)}{a}\right) = F_X\left(\frac{(y-b)}{a}\right)$ if a > 0. Then, $f_Y(y) = F'_Y(y)$ =

$$F_X'\left(\frac{(y-b)}{a}\right) = \frac{1}{a}f_x\left(\frac{(y-b)}{a}\right) = \frac{1}{a\sigma\sqrt{2\pi}}e^{-\frac{\left(\left(\frac{y-b}{a}\right)-\mu\right)^2}{2\sigma^2}} = \frac{1}{a\sigma\sqrt{2\pi}}e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}} = \frac{1}{a\sigma\sqrt{2\pi}}e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}}.$$

Comparing with the standard definition of a Gaussian, we see that the parameters of Y are $(a\mu+b, (a\sigma)^2)$. A similar calculation holds if $a < 0$.

13.0 Exponential distribution

Suppose that customers arrive to a bank with an exponentially distributed inter-arrival time with mean 5 minutes. A customer walks into the bank at 3pm. What is the probability that the next customer arrives no sooner than 3:15?

Solution: We have $1/\lambda$=5. We need to compute 1-F(15) = $1-(1-e^{-\lambda x}) = e^{\frac{-15}{5}} = e^{-3} = 4.85\,\%$.

14.0 Exponential distribution

It is late August and you are watching the Perseid meteor shower. You are told that that the time between meteors is exponentially distributed with a mean of 200 seconds. At 10:05 pm, you see a meteor, after which you head to the kitchen for a bowl of icecream, returning outside at 10:08pm. How long do you expect to wait to see the next meteor?

Solution: Because the exponential distribution is memoryless, the expected waiting time is the same, i.e. 200 seconds, no matter how long your break for icecream. Isn't that nice?

15.0 Power law

Consider a power-law distribution with $x_{min} = 1$ and $\alpha = 2$ and an exponential distribution with $\lambda = 2$. Fill in the following table:

x	$f_{power\_law}(x)$	$f_{exponential}(x)$
1		
5		
10		
50		
100		

Solution:

x	$f_{power\_law}(x)$	$f_{exponential}(x)$
1	1	0.27
5	0.04	$9.07*10^{-5}$
10	0.01	$4.1*10^{-9}$
50	$4*10^{-4}$	$7.44*10^{-44}$
100	$1*10^{-4}$	$2.76*10^{-87}$

It should now be obvious why a power-law distribution is called 'heavy-tailed'!

16.0 Markov's inequality

Consider a random variable X that exponentially distributed with parameter $\lambda = 2$. What is the probability that $X > 10$ using (a) the exponential distribution (b) Markov's inequality.

Solution: (a) We need $1-F(10) = e^{-20} = 2.06*10^{-9}$. (b) The mean of this distribution is 1/2. So, $P(X \geq 10) \leq \frac{0.5}{10} = 0.05$. It is clear that the bound is very loose.

17.0 Joint probability distribution

Consider the following probability mass function defined jointly over the random variables, X, Y, and Z:

P(000) = 0.05; P(001) = 0.05; P(010) = 0.1; P(011)=0.3;P(100) = 0.05; P(101) = 0.05; P(110) = 0.1; P(111)=0.3. (a) Write down $p_X, p_Y, p_z, p_{XY}, p_{XZ}, p_{YZ}$. (b) Are *X* and *Y*, *X* and *Z*, or *Y* and *Z* independent? What is the probability that *X=0* given that *Z=1*?

Solution: (a) p_X = {0.5, 0.5}; p_Y = {0.2, 0.8}; p_Z = {0.3, 0.7}; p_{XY} = {0.1, 0.4, 0.1, 0.4}; p_{XZ} = {0.15, 0.35, 0.15, 0.35}; p_{YZ} = {0.1, 0.1, 0.2, 0.6}

(b) *X* and *Y* are independent because $p_{XY} = p_X p_Y$. *X* and *Z* are independent because $p_{XZ} = p_X p_Z$.

(c) P(*X*=0|*Z*=1) = P(*X*=0 AND *Z*=1)/P(*Z*=1) = 0.35/0.7 = 0.5.