CS 798: Homework Assignment 3 (Queueing Theory)

Assigned: October 6, 2009

1.0 Little's law

Patients arriving to the emergency room at the Grand River Hospital have a mean waiting time of three hours. It has been found that, averaged over the period of a day, that patients arrive at the rate of one every five minutes. (a) How many patients are awaiting treatment on average at any given point in time? (b) What should be the size of the waiting room so that it can accommodate everyone?

2.0 A stochastic process

Consider that in Example 4, a person is on an infinite staircase on stair number 10 at time 0 and potentially moves once every clock tick. Suppose that he moves from stair i to stair $i+1$ with probability 0.2, and from stair i to stair $i-1$ with probability 0.2 (the probability of staying on stair i is 0.6). Compute the probability that the person is on each stair at time 1 (after the first move), time 2, and time 3.

3.0 Discrete and continuous state and time processes

Come with your own examples for all four combinations of discrete state/discrete time/continuous state/continuous time processes.

4.0 Markov process

Is the process in Exercise 2 a Markov process? Why or why not?

5.0 Homogeneity

Is the process in Exercise 2 homogeneous? Why or why not?

6.0 Representation

(a) Represent the process in Exercise 2 using a transition matrix and a state transition diagram. (b) Do the rows in this matrix have to sum to 1? Do the columns in this matrix have to sum to 1? Why or why not? (c) Now, assume that the staircase has only 4 steps. Make appropriate assumptions (what are these?) to represent this finite process as a transition matrix and a state transition diagram.

7.0 Reducibility

Is the chain in Exercise 2 reducible? Why or why not?

8.0 Recurrence

Is state 1 in the chain in Exercise 6(c) recurrent? Compute f_1^1, f_1^2 and f_1^3.

9.0 Periodicity

Is the chain in Exercise 2 periodic? If not, give an example of a chain with period N for arbitrary $N > 1$.

10.0 Ergodicity

Is any state in the chain of Exercise 6(c) non-ergodic? Why or why not?

11.0 Stationary probability

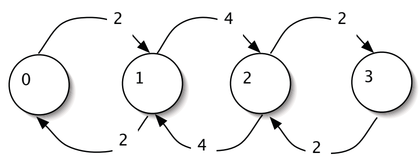
Compute the stationary probability distribution of the chain in Exercise 6(c)

12.0 Residence times

Compute the residence time in each state of the Markov chain in Exercise 6(c).

13.0 Stationary probability of a birth-death-process

Consider the state-rate-transition diagram shown below.



(a) Compare this with the state transition probability diagram in Exercise 6(c). What features are the same, and what differ?

(b) Write down the $\boldsymbol{Q}$ matrix for this system.

(c) Use the $\boldsymbol{Q}$ matrix to compute the stationary probability distribution of this chain.

14.0 Poisson process

Prove that the inter-departure time of a pure-death process is exponentially distributed.

15.0 Stationary probabilities of a birth-death process

Use Equation 30 to compute the stationary probability of the birth-death process in Exercise 13.

16.0 M/M/1 queue

Is the birth-death-process in Exercise 13 M/M/1? Why or why not?

17.0 M/M/1 queue

Consider a link to which packets arrive as a Poisson process at a rate of 450 packets/sec such that the time taken to service a packet is exponentially distributed. Suppose that the mean packet length is 250 bytes, and that the link capacity is 1 Mbps.

(a) What is the probability that the link's queue has 1, 2 and 10 packets respectively?

(b) What is the mean number of packets in the system? What is the mean number in the queue?

(c) What is the mean waiting time?

18.0 Responsive ($M/M/\infty$) server

Compute the ratio of P_j for a responsive server to the same value for an M/M/1 queue. How does this ratio behave as a function of j?

19.0 M/M/1/K server

Assume that the queueing system in Exercise 17 has 10 buffers. Compute an upper bound on the probability of packet loss.

20.0 M/D/1 queue

Compute the mean number of customers in an M/D/1 system that has a utilization of 0.9. (a) How does this compare with a similarly loaded M/M/1 system? (b) Compute the ratio of the mean number of customers as a function of ρ. (c) Use this to compare the behavior of an M/D/1 queue with that of an M/M/1 queue under heavy load.