CS 798: Homework Assignment 4 (Game Theory)

Assigned: October 28, 2009

1.0 Preferences

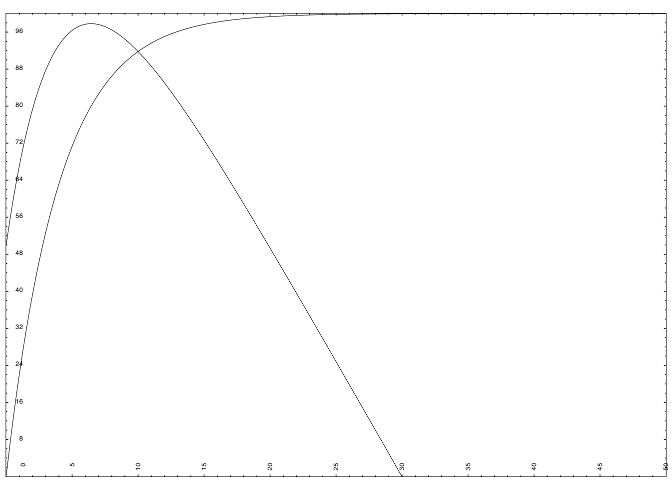
Suppose that you equally like a banana and a lottery that gives you an apple 30% of the time and a carrot 70% of the time. Also, you equally like a peach and a lottery that gives you an apple 10% of the time and a carrot 90% of the time. (a) What can you say about your relative preferences for bananas and peaches? If you had a lottery whose payoffs were bananas and carrots, what probability of winning a banana or a carrot would be equally preferred to a peach?

Solution: Denote apple = A, banana = B, carrot = C, peach = P. We are free to choose utilities as we wish, so let U(A)=0, U(C) = 1. Then, U(B) = .7 and U(P) =.9, so you prefer peaches to bananas. (b) Let P(win B) = p. Then $.7p + 1(1-p) = .9$, so $.3p = .1$, so $p = 0.33$.

2.0 Utility functions

Your cable company gives you 10GB of free data transfer a month, and charges $5/GB thereafter. Suppose that your utility from transferring x GB of data is $100(1-e^{-0.25x})$ and that your disutility from paying $1 is 1. How much data should you transfer in a month to maximize your utility?

Solution: Your net utility from transferring x GB is $100(1-e^{-0.25x})$ if $x<10$ and $100(1-e^{-0.25x}) - 5(x-10)$ otherwise. The plot of these two functions is shown below:



It is clear that the maximum occurs at x=10 for a value of approximately 92. So, your utility is maximized by transferring exactly 10GB/month.

3.0 Pure and mixed strategies

Consider the game of tic-tac-toe. What are the possible actions for the first move of the first player (ignore symmetries)? What would constitute a pure strategy? What would constitute a mixed strategy? Would you ever play a mixed strategy for this game? Why or why not?

Solution: The only possible first actions are: play corner, play middle, and play center. Depending on which move is played, the second player would have response, and depending on that response the first player would have a response etc. A pure strategy for each player is each valid response to the prior move (whether or not it is rational). A mixed strategy would play one of the pure strategies (i.e the entire sequence) with some probability. It turns out that in tic-tac-toe, with two expert players, a tie is guaranteed with a pure strategy, but a mixed strategy (depending over what you mix) could lose when played against an optimal strategy. So, it never makes sense to mix. In general, every component of a mixed strategy must be a potentially winning strategy. Otherwise, the mixed strategy would improve by discarding a component that can never win.

4.0 Zero-sum game

If the payoffs (a, $-a$) of every outcome of a zero sum game were changed so that the new payoffs were (a+5, -5a), the game would no longer be zero sum. But, would the structure of the game change?

Solution: No, because utilities are only unique to a affine transformation.

5.0 Representation

Represent the Pricing game of Example 7 in Normal form.

Solution:

	L	M	H
Y	(1,a-1)	(2,a-2)	(3,a-3)
N	(0,0)	(0,0)	(0,0)

6.0 Representation

Prove that normal and extensive form are equivalent if information sets are permitted.

Solution: We need to prove two things (a) if information sets are permitted every normal form game can be represented in extensive form (b) if information sets are permitted every extensive-form game can be represented in normal form. To prove (a): given a normal form game with n players, simply draw a tree of depth n, where all moves by the first player are associated with a node with an edge leading from the root to that node, and all nodes are in the same information set. Then, from each such node, draw an edge for each possible move for the second player, and place each set of nodes in the same information set. Repeat for each successive player, and label the leaves with the payoff from the corresponding array element. To prove (b): given the extensive form game, form paths from the root to each leaf. Decompose the path into moves by each of the players and find all possible moves by each player each time it is allowed to make a move. Let S_i^t denote the set of moves that player i can move on its t turn. Then the strategy space for player i is the cross product of these sets. Finally, the normal form is an n-dimensional matrix with the ith dimension indexed by the strategy space of the ith player, and the corresponding element having the payoff for these strategies.

7.0 Best response

What is the best response for the customer in the Pricing game (Example 7)?

Solution: The best response depends on the value of a. For each of the strategies of the ISP, i.e., L, M, and H, the best response is Y if *a-price* > 0, otherwise it is N.

8.0 Dominant strategy

Suppose that you are not well prepared for a final, and you think you might fail it. If you miss the exam, you will certainly fail it. What is your dominant strategy: attend or miss? Why?

Solution: If you attend, your payoff is your utility for either pass or fail, but if you miss, your payoff is your utility for fail. Assuming that utility(pass) > utility(fail), your payoff for attending is as good as or better than the payoff for not attending. So, your dominant strategy is to attend.

9.0 Bayesian game

Does the Bayesian game in Example 11 have a dominant strategy for the Row player? If so, what is it?

Solution:

It is easy to verify that no matter the type of the Column player (strong or weak signal), the best response for Row if Column plays S is D and if Column plays D is S. Therefore, knowing the type of the Column player does not help Row, and the game does not have a dominant strategy for Row.

10.0 Repeated game

Suppose that both players in Prisoner's Dilemma (Example 15) play their dominant strategy in an infinitely repeated game with a discount factor of 0.6. What is their payoff for the repeated game?

Solution: The one shot payoff is -3 for each, so the repeated payoff is $-3^* \sum_{i=0}^{\infty} 0.6^i = -3/.4 = -7.5$.

11.0 Dominant strategy equilibrium

Interpret the meaning of the dominant strategy equilibrium of Example 14. Look up how the 802.11e EDCA protocol solves this problem.

Solution:

It is dominant for both players to send rather than wait. In equilibrium, they always send right away so their packets always collide, and in fact, no progress is made, so that delays are actually infinite. This game illustrates the aphorism: haste makes waste. The EDCA protocol allows higher priority (delay sensitive) stations to wait for a shorter time than lower-priority stations before accessing the medium, therefore making it more probable that they would get access to medium and experience a shorter delay.

12.0 Iterated deletion

Show an example of a pure strategy that is dominated by a mixture of other pure strategies, although none of the strategies in the mixture dominate the pure strategy.

Solution: Consider the following game, where we only show the payoffs for Row:

	C1	C2
R1	0	0
R2	1	-1
R3	-2	2

Neither R2 nor R3 dominate R1. However any mixed strategy of R2 and R3 that plays R3 with a probability greater than 2/3 dominates R1. Therefore, we can delete R1 from the game.

13.0 Maximin

What are the maximin equilibria in Examples 10 and 14?

Solution: In Example 10, Row can get as low as -1 with S, but at least 0 with D, so its maximin strategy is D. Column is assured 1 with S, so its maximin strategy is S, and the equilibrium is DS.

In Example 14, Row maximizes its minimum payoff with S. The game is symmetric, so the maximin equilibrium is SS.

14.0 Maximin in a zero-sum game

Show that in Example 18 if Row uses any value of p other than 0.5, then it may get a payoff lower than 2.5 if Column plays either pure or mixed strategies.

Solution: In Figure 3, note that when p is smaller than 0.5, the column player can play pure strategy C1 to reduce Row's payoff below 2.5. Similarly, if p is greater than 0.5, Column can use a pure strategy C2 to reduce Row's payoff. For any value of p, Column can play a mixture qC1 + (1-q) C2 to give Row a payoff of $q(p+2) + (1-q)(4-3p)$. To make this smaller than 2.5, we set $q(p+2) + (1-q)(4-3p) < 2.5$, or $q > (3-6p)/(4-8p)$. For instance, if p=0, $q > 3/4$, and if p=1, q>3/4. (The inequality is not valid when p= 0.5.)

15.0 Nash equilibrium

Referring to Example 19, assume that if Column plays a mixed strategy with probability qH + (1-q)T instead of its Nash equilibrium strategy. What is Row's mixed strategy best response?

Solution: Let the row player play pH + (1-p)T. Then, its payoff, given Column's mixed strategy, is $p(q-(1-q))+(1-p)(-q+(1-q)) = 4pq -2q -2p +1 =(1-2p)(1-2q)$. If $q < 0.5$, p should be 0, otherwise p should be 1. Intuitively, if the Column player is more likely to play T, then Row should play T for sure and *vice versa*.

16.0 Correlated equilibrium

Does the WiFi game of Example 6 have a correlated equilibrium? If so, describe it.

Solution: Consider an external agency that tells the players to play pDS + (1-p)SD. When Row is told to play D, it knows that it will get a payoff of -1 if it deviates. Similarly, when told to play S, it will get 0 if it deviates (instead of 1). So, it will not deviate, independent of the value of p. By symmetry, the same analysis holds for Column, and therefore we have a correlated equilibrium. The external agency can arrange for any desired payoffs to Row and Column by adjusting p.

17.0 Price discrimination

Outline the design of a price-discrimination mechanism with n player types (whose valuations are known).

Solution: Assume that the valuations of each player are $v_1,...,v_n$ for minimum quantities of $q_1,...,q_n$. The scheme is essentially to charge v_i for q_i adjusting for the fact that player i could buy multiples of q_j $j<i$ if that minimizes its total cost.

18.0 VCG mechanism

The CIO of a company wants to decide how much capacity to buy from its ISP. The cost of capacity is \$20/Mbps/month. There are three departments in the company, who value capacity as follows: department 1 (D1) values capacity xMbps/month at \$20(*1-$e^{-0.5x}$), D2 values it at \$40(*1-$e^{-0.5x}$), D3 values it at \$80(*1-$e^{-0.5x}$). (a) Assuming the disutility of ISP payment is linear in the amount of payment, what is the overall function that the CIO should maximize? (b) What is type of each department? (c) What is the optimal social choice? (d) What are the Clarke Pivot payments for each department? (e) Is this budget balanced?

Solution: (a) The overall function is $(20+40+80)(1-e^{-0.5x}) - 20x = 140(1-e^{-0.5x}) - 20x$.

(b) The types are the only unknowns in the utility functions, i.e. 20, 40, and 80 respectively.

(c) The optimal social choice comes from maximizing the function in (a). Setting $f(x) = 140(1-e^{-0.5x}) - 20x$, solve for $f'(x^*)=0$, so that $x^* = 2.5055$.

(d) To compute x^{-1}, we maximize $(120(1-e^{-0.5x}) - 20x)$ to get 2.197. Similarly, $x^{-2} = 1.832$, and $x^{-3} = 0.8109$. Thus, $p_1 = v_2(x^{-1}) + v_3(x^{-1}) - (v_2(x^*) + v_3(x^*)) = (40+80)(1-e^{-0.5*2.197}) - (40+80)(1-e^{-0.5*2.5055}) = 120*(e^{-1.25275}-e^{-1.0985}) = -5.718$.

Similarly, $p_2 = v_1(x^{-2}) + v_3(x^{-2}) - (v_1(x^*) + v_3(x^*)) = 100(e^{-1.25275}-e^{-0.5*1.832}) = -11.439$,

$p_3 = v_1(x^{-3}) + v_2(x^{-3}) - (v_1(x^*) + v_2(x^*)) = 60(e^{-0.5*0.8109}-e^{-1.25275}) = 60*(e^{-1.25275}-e^{-0.4055}) = -22.857$.

(e) No, the budget is not balanced: the CIO has to pay each department.